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Minimal Approximations, Orbital Elementary Modules, and Orbit Algebras of Regular Modules

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This paper continues the study of the functor, introduced in [7]. Let $A = k\mathcal{Q}$ be a connected wild hereditary path-algebra, where $\mathcal{Q}$ is some finite quiver without oriented cycles and k is some field. The category of finitely generated left A-modules is denoted by A-mod, and maps are written opposite to the scalars. Assume A has at least three simple modules and let X be a quasi-simple regular module without self-extensions. The right perpendicular category $X^{\perp}$ of X is the full subcategory of A-mod, defined by the objects $\{M \mid \mathrm{Hom}_A(X, M) = 0 = \mathrm{Ext}^1_A(X, M)\}$. It is equivalent to a module category C-mod, where $C = k\mathcal{Q}_X$ is a connected wild hereditary algebra with $n - 1$ simple modules. Denote by C-reg the category of modules, regular in $X^{\perp} \cong C$-mod, by τ_C, respectively, τ_A the Auslander–Reiten translations in $X^{\perp}$, respectively, A-mod and by A-reg the category of regular A-modules. If $0 \to \tau_A X \to Z \to X \to 0$ is the Auslander–Reiten sequence, ending in X, then Z is a quasi-simple brick in C-reg, [13, 27]. The main result of [7] says, that there is a full and dense functor F: C-reg $\to A$-reg, with $F\tau_C \cong \tau_A F$ and with $F(M) = 0$ if and only if $M \in \mathrm{add}\{\tau_C^i Z \mid i \in \mathbb{Z}\}$. This functor also induces a bijection between the sets of regular Auslander–Reiten components $\Omega(C)$ of C and $\Omega(A)$ of A. As a consequence, for example, bijections between the sets of regular Auslander–Reiten components $\Omega(A_1)$ and $\Omega(A_2)$ for any two connected wild hereditary path-algebras $A_1 = k\mathcal{Q}^{(1)}$ and $A_2 = k\mathcal{Q}^{(2)}$ can be constructed. For the construction of this functor tilting theory was used.

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In [18], a different construction of the functor F was given, using universal filtrations. More precise, the functor F is defined in [18] as the composition $F = e_\infty h_\infty = h_\infty e_\infty$ of two functors e_∞, h_∞ and these two functors are constructed by universal filtrations.

Both constructions have in common that direct limits and inverse limits are used for the definition of F. The first aim of the paper is, to describe this functor without the use of limits. For this, minimal approximations are considered. Recall, that a map $f\colon M \to N$ is called right minimal, if $\alpha f = f$, with $\alpha \in \mathrm{End}_A(M)$ implies $\alpha \in \mathrm{Aut}(M)$, [2]. If $\mathscr{C} \subset A$-mod is a subcategory, a morphism $f\colon C \to M$, with $C \in \mathscr{C}$ is called a right $\mathscr{C}$-approximation, if for any $C' \in \mathscr{C}$ the induced map (C', f): $\mathrm{Hom}_A(C', C) \to \mathrm{Hom}_A(C', M)$ is surjective, [5, 3]. A right minimal right $\mathscr{C}$-approximation is called a minimal right $\mathscr{C}$-approximation. Minimal left $\mathscr{C}$-approximations are defined dually. Using the construction in [18], we get the following new description of e_∞, respectively, h_∞ and hence of $F = e_\infty h_\infty = h_\infty e_\infty$.

THEOREM. *Let $A = k\mathscr{Q}$ be connected wild hereditary, X a quasi-simple regular stone and $M \in C$-*reg.

(a) *The minimal right* $\mathrm{add}\{\tau_A X, \tau_C^i Z \mid i > 0\}$*-approximation* $\rho\colon K(M) \to M$ *is injective with cokernel $h_\infty(M)$.*

(b) *The minimal left* $\mathrm{add}\{X, \tau_C^{-i} Z \mid i > 0\}$*-approximation* $\lambda\colon M \to Q(M)$ *is surjective with kernel $e_\infty(M)$.*

(c) *The following diagram is commutative*

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K(M) \arrow[r, "\rho'"] \arrow[d, equal] & e_\infty(M) \arrow[r] \arrow[d] & F(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K(M) \arrow[r, "\rho"] & M \arrow[r] \arrow[d, "\lambda"] & h_\infty(M) \arrow[r] \arrow[d, "\lambda'"] & 0 \\
 & & Q(M) \arrow[r, equal] \arrow[d] & Q(M) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

Moreover, the mono $\rho'\colon K(M) \to e_\infty(M)$ is a minimal right $\mathrm{add}\{\tau_A X, \tau_C^i Z \mid i > 0\}$*-approximation, the epi $\lambda'\colon h_\infty(M) \to Q(M)$ is a minimal left* $\mathrm{add}\{X, \tau_C^{-i} Z \mid i > 0\}$*-approximation.*

This might be considered as a mainly technical result, but it has a surprising consequence. It creates a new class of elementary modules, see 2.3, called orbital elementary. Recall that a regular module E is called elementary, if there is no short exact sequence $0 \to U \to E \to V \to 0$, with U, V both nonzero and regular, [22, 17]. An indecomposable regular module E is called orbital elementary, if each short exact sequence $0 \to U \to \overline{E} \to V \to 0$ with U, V both regular and $\overline{E} \in \operatorname{add}\{\tau_A^i E | i \in \mathbb{Z}\}$, splits. Orbital elementary modules have a quite startling property: If E is orbital elementary, we can get rather complete information on all maps between the modules in the τ_A-orbit $(\tau_A^i E | i \in \mathbb{Z})$.

The most convenient tool for the description of the maps between the modules $\tau_A^i M$ in the τ_A-orbit of a regular module M is the orbit algebra $\mathscr{O}(M)$, introduced in Section 3, and called in [20] the algebra of the endo-functor τ_A at M. The algebra $\mathscr{O}(M) = \bigoplus_{i \in \mathbb{Z}} \mathscr{O}_i(M)$ is a $\mathbb{Z}$-graded algebra, where $\mathscr{O}_i(M) = \operatorname{Hom}_A(M, \tau_A^i M)$ and multiplication is given by $\mathscr{O}_i(M) \times \mathscr{O}_j(M) \to \mathscr{O}_{i+j}(M)$, $(f, g) \mapsto f \cdot \tau_A^i g$, where $\cdot$ denotes the composition of maps.

If E is orbital elementary with $\operatorname{End}_A(E) = k$, then the orbit algebra $\mathscr{O}(E)$ is a free k-algebra $k\langle X_i^{(j)} | 1 \leq i,\ 1 \leq j \leq r_i \rangle$ in infinitely many variables $X_i^{(j)} \in \mathscr{O}_i(E)$. Moreover is the power series $\sum_{i \geq 1} r_i t^i$ a rational function, see 5.4, 5.5.

Additionally it is discussed in paragraph four, when the orbit algebra $\mathscr{O}(M)$ of a regular module $M \neq 0$ is finitely generated as an algebra and examples are given. It is an open problem, whether there exist orbit algebras $\mathscr{O}(M)$ of regular modules M which are neither finitely generated nor free.

For basic results we refer to the books [4, 25], for special results on wild hereditary algebras to the survey [16].

1. MINIMAL APPROXIMATIONS

1.1. The concept of minimal approximations, introduced in [2, 5, 3] is central for this paper. If $\mathscr{C}$ is a subcategory of A−mod, a map $\rho: C \to M$ with $C \in \mathscr{C}$ is called a *right $\mathscr{C}$-approximation,* if for all $C' \in \mathscr{C}$ the induced map (C', ρ): $\operatorname{Hom}(C', C) \to \operatorname{Hom}(C', M)$ is surjective. The map ρ is called *right minimal,* if $c\rho = \rho$, with $c \in \operatorname{End}_A(C)$ implies c is an automorphism, or equivalently, if no nonzero direct summand of C is in the kernel of ρ, see [4]. A right minimal right $\mathscr{C}$-approximation is called a *minimal right $\mathscr{C}$-approximation.* It always is unique up to an isomorphism, see [4]. In the dual way *minimal left $\mathscr{C}$-approximations* are defined.

A right, respectively, left $\mathscr{C}$-approximation f: $C \to M$, respectively, g: $M \to C$ always exists, if there are only finitely many indecomposables C_i in $\mathscr{C}$ with $\operatorname{Hom}_A(C_i, M) \neq 0$, respectively, $\operatorname{Hom}_A(M, C_i) \neq 0$. If X is indecomposable with $\operatorname{End}(X) = k$, minimal right add X-approximations can be constructed easily. If M is any module and $f_1, \dots, f_r$ is a k-basis of

$\mathrm{Hom}_A(X, M)$, then $\rho = (f_1, \dots, f_r)^t\colon X^r \to M$ is a minimal right add X-approximation of M. Frequently we call ρ a *right universal map* for $\mathrm{Hom}(X, M)$ in this case. Dually one constructs the *left universal map* for $\mathrm{Hom}(M, X)$.

1.2. If $(\mathscr{T}, \mathscr{F})$ is a *torsion pair* in A-mod, with torsion class $\mathscr{T}$ and torsion-free class $\mathscr{F}$, and M is any A-module, there exists the *canonical short exact sequence*

$$0 \to tM \xrightarrow{\epsilon} M \xrightarrow{\pi} fM \to 0,$$

with $M \in \mathscr{T}$ and $fM \in \mathscr{F}$. The inclusion ϵ is a minimal right $\mathscr{T}$-approximation of M, whereas π is a minimal left $\mathscr{F}$-approximation of M in this case.

A module $T \in \mathscr{T}$ is called Ext-projective in $\mathscr{T}$, if $\mathrm{Ext}^1_A(T, M) = 0$ for all $M \in \mathscr{T}$. Dually $F \in \mathscr{F}$ is called Ext-injective in $\mathscr{F}$, if $\mathrm{Ext}^1_A(N, F) = 0$ for all $N \in \mathscr{F}$. It was shown in [11], that a module T was Ext-projective in $\mathscr{T}$ if and only if $\tau_A T$ was Ext-injective in $\mathscr{F}$.

Finally it should be mentioned, that for an indecomposable module $M \in \mathscr{T}$, not Ext-projective, there always exists a relative Auslander−Reiten sequence $0 \to \tau_{\mathscr{T}} M \to E \to M \to 0$ in $\mathscr{T}$, ending in M, but not necessarily a relative Auslander−Reiten sequence, starting in M, even if M is not Ext-injective, see for example, [1]. The dual statements hold for the torsion-free class $\mathscr{F}$.

1.3. Let $A = k\mathscr{Q}$ connected wild hereditary with $n > 2$ simple modules and X be a quasi-simple regular stone. The word stone denotes an indecomposable module without self-extensions. The *right perpendicular category* $X^{\perp}$ of X, respectively, the *left perpendicular category* ${}^{\perp}X$ of X is the full subcategory of A-mod with objects $\{M | \mathrm{Hom}_A(X, M) = 0 = \mathrm{Ext}^1_A(X, M)\}$, respectively, $\{N | \mathrm{Hom}_A(N, X) = 0 = \mathrm{Ext}^1_A(N, X)\}$. Since X is not projective, $X^{\perp} = {}^{\perp}(\tau_A X)$ follows from the Auslander−Reiten formula. Moreover we have $X^{\perp} \cong C$-mod, where C is a connected wild hereditary algebra with $n - 1$ simple modules [27]. Especially there exist Auslander−Reiten sequences in $X^{\perp}$, and we write τ_C, τ_C^- for the Auslander−Reiten translations in $X^{\perp}$. The preprojective component of $X^{\perp}$ is denoted by $\mathscr{P}(C)$, its preinjective component by $\mathscr{I}(C)$.

If $0 \to \tau_A X \to Z \to X \to 0$ is the Auslander−Reiten sequence, ending in X, then Z is a brick with $\mathrm{End}_A(Z) = k$, by [13], and it is a quasi-simple regular module in $X^{\perp}$, by [27]. The name Z is reserved for this specific module.

1.4. Following [18, 2.4], for a quasi-simple regular stone X we denote by $(\mathscr{E}_X, \mathscr{F}_X)$ the torsion pair with torsion class $\mathscr{E}_X = \{M | \mathrm{Ext}(X, M) = 0\}$ and torsion-free class $\mathscr{F}_X$, the class of modules, cogenerated by $\tau_A X$. If P is the minimal projective generator of $X^{\perp}$, then $T' = X \oplus P$ is a tilting module with $\mathscr{E}_X = \mathrm{Gen}(T')$, [7]. We have $X^{\perp} \subset \mathscr{E}_X$, and the relative

Auslander–Reiten quiver $\Gamma(\mathscr{E}_X)$ has exactly one preprojective component, namely, the preprojective component $\mathscr{P}(C)$ of $X^{\perp}$ and exactly one preinjective component, the preinjective component $\mathscr{I}(A)$ of A-mod, see, for example, [7].

Dually, we denote by $(\mathscr{G}_X, \mathscr{H}_X)$ the torsion pair with torsion-free class $\mathscr{H}_X = \{M | \operatorname{Hom}_A(X, M) = 0\}$. The modules generated by X constitute the torsion class $\mathscr{G}_X$. If I denotes the minimal injective cogenerator of $X^{\perp} = {}^{\perp}(\tau_A X)$, then $T = I \oplus \tau_A X$ is a cotilting module, and $\mathscr{H}_X$ is the torsion-free class of T, that is $\mathscr{H}_X = \{M | \operatorname{Ext}^1_A(M, T) = 0\}$, [18, 2.1]. We have $X^{\perp} \subset \mathscr{H}_X$ and the preprojective component $\mathscr{P}(A)$ of A-mod is the preprojective component of the relative Auslander–Reiten quiver $\Gamma(\mathscr{H}_X)$ of $\mathscr{H}_X$, whereas $\mathscr{I}(C)$ is its unique preinjective component.

If M is indecomposable in $\mathscr{H}_X$, not in $\mathscr{I}(C)$ and N is preinjective in $X^{\perp}$, then $\operatorname{Hom}_A(N, M) = 0$, since $\mathscr{I}(C)$ is successor-closed in $\mathscr{H}_X$. Therefore $\mathscr{H}_X^{<}$, the class of objects of $\mathscr{H}_X$ without nonzero direct summand in $\mathscr{I}(C)$ is a torsion-free class, too. Dually is $\mathscr{E}_X^{>}$, the class of objects in $\mathscr{E}_X$ without nonzero direct summand in $\mathscr{P}(C)$ a torsion class.

We have $\mathscr{E}_X \cap \mathscr{H}_X = X^{\perp}$ and $\mathscr{E}_X^{>} \cap \mathscr{H}_X^{<} = C$-reg, where C-reg is the category of modules, regular in $X^{\perp}$. All the modules $\tau_C^i Z$ are in $\mathscr{E}_X^{>} \cap \mathscr{H}_X^{<}$.

1.5. We consider all the torsion pairs $(\mathscr{G}_j, \mathscr{H}_j)$, for $j \geq 0$, where $\mathscr{H}_j = \mathscr{H}_{\tau_A^j X}$ and $\mathscr{G}_j = \mathscr{G}_{\tau_A^j X}$. Take $M \in \mathscr{H}_j$ and consider the canonical short exact sequence

$$0 \to g_{\tau_A^{j+1}X} M \to M \xrightarrow{\phi_{\tau_A^{j+1}X}} h_{\tau_A^{j+1}X} M \to 0,$$

of M with respect to the torsion pair $(\mathscr{G}_{j+1}, \mathscr{H}_{j+1})$. Since $g_{\tau_A^{j+1}X} M \in \mathscr{G}_{j+1}$, it is generated by $\tau_A^{j+1}X$.

If ρ: $\tau_A^{j+1}X^r \to M$, with $r = \dim \operatorname{Hom}_A(\tau_A^{j+1}X, M)$, is the right universal map, we get $g_{\tau_A^{j+1}X} M = \operatorname{Im} \rho$.

It was shown in [18, 3.1] that for $M \in \mathscr{H}_j$ indecomposable, the right universal map ρ: $\tau_A^{j+1}X^r \to M$ was injective or surjective. If it is surjective, then M is Ext-injective in $\mathscr{H}_j$. For $M \in \mathscr{H}_j^{<}$ it always is injective. Furthermore, it was shown in [18, 3.4], that

$$h_{\tau_A^{j+1}X} : \mathscr{H}_j \to \mathscr{H}_{j+1}$$

defines a full and dense functor, which respects epimorphisms. For an indecomposable $N \in \mathscr{H}_j$ we have $h_{\tau_A^{j+1}X}(N) = 0$ only if either $N = \tau_A^{j+1}X$ or N is injective in $(\tau_A^j X)^{\perp}$, [18].

Additionally it follows from [18, 4.5] that the restriction of $h_{\tau_A^{j+1}X}$ to $\mathscr{H}_j^<$ has image $\mathscr{H}_{j+1}^<$, hence $h_{\tau_A^{j+1}X}$ induces a full and dense functor

$$h_{\tau_A^{j+1}X} : \mathscr{H}_j^< \to \mathscr{H}_{j+1}^<,$$

with $h_{\tau_A^{j+1}X}(N) = 0$ if and only if $N \in \operatorname{add} \tau_A^{j+1}X$.

Let $h_j = h_{\tau_A^j X} h_{\tau_A^{j-1}X} \cdots h_{\tau_A X}: \mathscr{H}_0^< \to \mathscr{H}_j^<$ be the composition. Then we get from [18].

LEMMA. (a) *The functor $h_j: \mathscr{H}_0^< \to \mathscr{H}_j^<$ is full and dense and respects epis.*

(b) *Let $M \in \mathscr{H}_0^<$ be indecomposable with $h_j(M) = 0$. Take $1 \le i \le j$ minimal with $h_i(M) = 0$. Then $i = 1$ if and only if $M = \tau_A X$ and $i > 1$ if and only if $M \cong \tau_C^{i-1}Z$.*

1.6. If M is a module with a filtration $0 \subset M_1 \subset \cdots \subset M_{m-1} \subset M_m = M$ and $M_i/M_{i-1} \cong Y_i$ for $1 \le i \le m$, we write

$$M : Y_1|Y_2| \cdots |Y_m.$$

If additionally $Y_i = X_i^{r_i}$ for $1 \le i \le m-1$ with X_i indecomposable such that the embedding $X_i^{r_i} \to M/M_{i-1}$ is right universal for $\operatorname{Hom}_A(X_i, M/M_{i-1})$, we write

$$M : X_1^{r_1}|X_2^{r_2}| \cdots |X_{m-1}^{r_{m-1}}|| \frac{M}{M_{m-1}},$$

and call this filtration *ascending* Hom-*universal*. The *descending* Hom-*universal* filtrations are defined dually. For more details see [18, 4.2, 4.3].

1.7. For $M = M_0 \in \mathscr{H}_0^<$ indecomposable with $h_j(M) = M_j \neq 0$ we can construct M_j from M by a sequence ρ_{i+1} with $0 \le i < j$ of injective right universal maps

$$0 \to \tau_A^{i+1} X^{r_{i+1}} \xrightarrow{\rho_{i+1}} M_i \xrightarrow{\phi_{i+1}} M_{i+1} \to 0.$$

Consequently M has an ascending Hom-universal filtration

$$M : \tau_A X^{r_1}|\tau_A^2 X^{r_2}| \cdots |\tau_A^j X^{r_j}||M_j.$$

The surjective maps ϕ_j are natural and so is the composition $\phi_{(j)}: M_0 \to M_j$. Let $K_j(M)$ be the kernel of $\phi_{(j)}$, with $K_j(M) = M_0$, if $M_j = 0$. Hence there is a short exact sequence

$$0 \to K_j(M) \xrightarrow{\epsilon} M \xrightarrow{\phi_{(j)}} h_j(M) \to 0.$$

PROPOSITION. *Let $M \in \mathscr{H}_0^<$.*

(a) *For $j \ge 1$ we get $K_j(M) = \tau_A X^t \otimes \tau_C Z^{t_1} \oplus \cdots \oplus \tau_C^{j-1} Z^{t_{j-1}}$.*

(b) *The inclusion* $\epsilon\colon K_j(M) \to M$ *is a minimal right* $\operatorname{add}(\tau_A X \oplus (\bigoplus_{1\le i<j} \tau_C^i Z))$*-approximation of* M.

(c) *The induced inclusion* $\epsilon'\colon \tau_C Z^{t_1} \oplus \cdots \oplus \tau_C^{j-1} Z^{t_{j-1}} \hookrightarrow M$ *is a minimal right* $\operatorname{add} \bigoplus_{i=1}^{j-1} \tau_C^i Z$*-approximation of* M.

Proof. (a) If $M_j = 0$, the claim follows from 1.5(b), so assume $M_j \neq 0$. Since $\mathscr{H}_0^<$ is a torsion-free class, the short exact sequence

$$(\eta) \quad 0 \to K_j(M) \to M \to M_j \to 0$$

implies $K_j(M) \in \mathscr{H}_0^<$. Since M has an ascending Hom-universal filtration

$$M : \tau_A X^{r_1} | \tau_A^2 X^{r_2} | \cdots | \tau_A^j X^{r_j} || M_j,$$

we know that $K_j(M)$ has a filtration

$$K_j(M) : \tau_A X^{r_1} | \tau_A^2 X^{r_2} | \cdots | \tau_A^{j-1} X^{r_{j-1}} | 0.$$

We see by induction that this filtration is Hom-universal. For $i = 1$ we get $r_1 \le \dim \operatorname{Hom}_A(\tau_A X, K_j(M))$, from the filtration. Applying $\operatorname{Hom}_A(\tau_A X, -)$ to (η) gives $0 \to \operatorname{Hom}(\tau_A X, K_j(M)) \to \operatorname{Hom}(\tau_A X, M)$. Hence $\dim \operatorname{Hom}_A(\tau_A X, K_j(M)) \le \dim \operatorname{Hom}_A(\tau_A X, M) = r_1$ and the inclusion $\tau_A X^{r_1} \hookrightarrow K_j(M)$ is right universal. We get the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \tau_A X^{r_1} \arrow[d] & \tau_A X^{r_1} \arrow[d] & & \\
0 \arrow[r] & K_j(M) \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M_j \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (K_j(M))_1 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & M_j \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

and $(K_j(M))_1$ has a filtration

$$\left(K_j(M)\right)_1 : \tau_A^2 X^{r_2} | \tau_A^3 X^{r_3} | \cdots | \tau_A^j X^{r_j} | 0.$$

Suppose by induction that for $1 \le i < j$:

$$0 \to \left(K_j(M)\right)_i \to M_i \to M_j \to 0$$

is exact and $(K_j(M))_i$ has a filtration

$$\left(K_j(M)\right)_i : \tau_A^{i+1} X^{r_{i+1}} | \cdots | \tau_A^j X^{r_j} | 0.$$

Similar to the case $i = 1$ we get $\dim \operatorname{Hom}(\tau_A^{i+1}X, (K_j(M))_i) = r_{i+1}$, hence the inclusion $\tau_A^{i+1}X^{r_{i+1}} \to (K_j(M))_i$ is right universal and there is a short exact sequence $0 \to (K_j(M))_{i+1} \to M_{i+1} \to M_j \to 0$.

Consequently we have $h_j(K_j(M)) = 0$ and the claim follows from 1.5.

(b) If $h_j(M) = 0$, then $M = \tau_A X^t \oplus \tau_C Z^{t_1} \oplus \cdots \oplus \tau_C^{j-1} Z^{t_{j-1}}$ by 1.5 and obviously is the identity map $K_j(M) = M \to M$ a minimal right $\operatorname{add}(\tau_A X \oplus (\bigoplus_{i \le i < j} \tau_C^i Z))$-approximation of M.

Let $h_j(M)$ be nonzero and $K_j(M) = \tau_A X^t \oplus \overline{Z}$ with $\overline{Z} = \tau_C Z^{t_1} \oplus \cdots \oplus \tau_C^{j-1} Z^{t_{j-1}}$. Further let $K_1 \in \operatorname{add}(\oplus \tau_A X(\bigoplus_{1 \le i < j} \tau_C^i Z))$ and $f\colon K_1 \to M$.

We have $h_j(K_1) = 0$ by 1.5, hence we get the following commutative diagram

$$\begin{array}{ccccccc} & & K_1 & \longrightarrow & 0 & = & h_j(K_1) \\ & & \downarrow f & & & & \downarrow h_j(f) \\ 0 \longrightarrow K_j(M) & \xrightarrow{\epsilon} & M & \longrightarrow & & & h_j(M) \longrightarrow 0. \end{array}$$

Hence f factors through ϵ and ϵ is a right $\operatorname{add}(\tau_A X \oplus (\bigoplus_{i \le i < j} \tau_C^i Z))$-approximation. It is minimal, since ϵ is injective.

(c) Take $Z_1 \in \operatorname{add} \bigoplus_{1 \le i < j} \tau_C^i Z$ and $g\colon Z_1 \to M$. As in part (b) we get $h_j(Z_1) = 0$ and g factors through ϵ. But $Z_1 \in X^{\perp}$, therefore $\operatorname{Hom}_A(Z_1, \tau_A X) = 0$, hence g has a factorisation through the inclusion ϵ': $\bigoplus_{i=1}^{j-1} \tau_C^i Z^{t_i} \to M$. Therefore ϵ' is a right $\operatorname{add} \bigoplus_{i=1}^{j-1} \tau_C^i Z$-approximation. Since it is injective, it is a minimal right approximation.

1.8. Dually to the above procedure, we consider the family of torsion pairs $(\mathscr{E}_i, \mathscr{F}_i)$, with torsion class $\mathscr{E}_i = \mathscr{E}_{\tau_A^{-i}X}$ and torsion-free class $\mathscr{F}_i = \mathscr{F}_{\tau_A^{-i}X}$. Additionally we consider the torsion classes $\mathscr{E}_i^{>}$ of modules in $\mathscr{E}_i$ without nonzero direct summand in $\mathscr{P}((\tau_A^{-i}X)^{\perp})$, the preprojective component of $(\tau_A^{-i}X)^{\perp}$. For $M \in \mathscr{E}_i^{>}$ we consider the canonical short exact sequence

$$0 \to e_{\tau_A^{-i-1}X}(M) \to M \to f_{\tau_A^{-i-1}X}(M) \to 0,$$

with $e_{\tau_A^{-i-1}X}(M) \in \mathscr{E}_{i+1}$ and $f_{\tau_A^{-i-1}X}(M) \in \mathscr{F}_{i+1}$.

Dually to 1.5, $e_{\tau_A^{-i-1}X}$ defines a full and dense functor $e_{\tau_A^{-i-1}X}\colon \mathscr{E}_i \to \mathscr{E}_{i+1}$ which respects monos and by restriction a full and dense functor $e_{\tau_A^{-i-1}X}\colon \mathscr{E}_i^{>} \to \mathscr{E}_{i+1}^{>}$. For $M \in \mathscr{E}_i^{>}$, the module $f_{\tau_A^{-i-1}X}(M)$ is in $\operatorname{add} \tau_A^{-i}X$ and the epi $\lambda_i\colon M \to f_{\tau_A^{-i-1}X}(M) = \tau_A^{-i}X^{s_i}$ is a minimal left $\operatorname{add} \tau_A^{-i}X$-approximation of M. For $M \in \mathscr{E}_i^{>}$ we get $e_{\tau_A^{-i-1}X}(M) = 0$ if and only if $M \in \operatorname{add} \tau_A^{-i}X$.

If $e_i\colon \mathscr{E}_0 \to \mathscr{E}_i$ denotes the composition of the functors $e_{\tau_A^{-j}X}$, it induces by restriction a functor $e_i\colon \mathscr{E}_0^{>} \to \mathscr{E}_i^{>}$. The following is the dual version of 1.5, [18, 3.6].

LEMMA. (a) *$e_i\colon \mathscr{E}_0^{>} \to \mathscr{E}_i^{>}$ is a full and dense functor which respects monos.*

(b) *Let $M \in \mathscr{E}_0^{>}$ be indecomposable with $e_i(M) = 0$. If $1 \le j \le i$ is minimal with $e_j(M) = 0$, then $M \cong X$ for $j = 1$ and $M \cong \tau_C^{1-j}Z$ for $j > 1$.*

1.9. For ${}_0M = M \in \mathscr{E}_0^{>}$ with $e_i(M) \neq 0$, we construct ${}_iM = e_i(M)$ by a sequence of surjective left universal maps λ_j,

$$0 \to {}_{j+1}M \to {}_jM \xrightarrow{\lambda_j} \tau_A^{-j}X^{s_j} \to 0.$$

Consequently ${}_0M$ has a descending Hom-universal filtration

$${}_0M : {}_iM || \tau_A^{1-i}X^{s_{1-i}}| \cdots |\tau_A^{-}X^{s_1}|X^{s_0}.$$

For ${}_0M = M \in \mathscr{E}_0^{>}$ with $e_i(M) \neq 0$ we get short exact sequences

$$0 \to {}_iM \to M \to Q_i(M) \to 0.$$

For $e_i(M) = 0$ we define $Q_i(M) = M = {}_0M$. Dually to 1.6 we have.

PROPOSITION. *Let $M \in \mathscr{E}_0^{>}$.*

(a) *For $i > 0$ we get $Q_i(M) = X^r \oplus (\bigoplus_{j=1}^{i-1} \tau_C^{-j}Z^{r_j})$.*

(b) *The epi $\lambda\colon M \to Q_i(M)$ is a minimal left* $\operatorname{add}(X \oplus (\bigoplus_{j=1}^{i-1}\tau_C^{j}Z))$*-approximation of M.*

(c) *The induced projection $\lambda'\colon M \to \bigoplus_{j=1}^{i-1}\tau_C^{-j}Z^{r_j}$ is a minimal left* add $\bigoplus_{j=1}^{i-1}\tau_C^{-j}Z$*-approximation of M.*

1.10. In strict analogy to [18] we define the functors $e_\infty\colon \mathscr{E}_0^{>} \to A$-mod and $h_\infty\colon \mathscr{H}_0^{<} \to A$-mod by

$$e_\infty(M) = \varprojlim(\cdots \to {}_2M \to {}_1M \to {}_0M = M) = e_m(M) \quad \text{for } m \gg 0,$$
$$h_\infty(M) = \varprojlim(M = M_0 \to M_1 \to M_2 \to \cdots) = h_m(M) \quad \text{for } m \gg 0.$$

The functor e_∞ (respectively, h_∞) is full, respects monos (respectively, epis) and has image those A-modules without nonzero preprojective (respectively, preinjective) direct summands, [18, 4.1, 4.3].

It follows further from [18] that the functor h_∞, restricted to $\mathscr{E}_0^{>} \cap \mathscr{H}_0^{<}$ has image $\mathscr{E}_0^{>} \cap \operatorname{Im} h_\infty$, that is the modules in $\mathscr{E}_0^{>}$ without nonzero A-preinjective direct summands. Dually, the restriction of e_∞ to $\mathscr{E}_0^{>} \cap \mathscr{H}_0^{<}$ has image the modules in $\mathscr{H}_0^{<}$ without indecomposable A-preprojective direct summands. Hence we can consider the compositions $e_\infty h_\infty\colon \mathscr{E}_0^> \cap \mathscr{H}_0^< \to A$-mod and $h_\infty e_\infty\colon \mathscr{E}_0^> \cap \mathscr{H}_0^< : \to A$-mod. It was the main result of [18], that

(1) Both functors $e_\infty h_\infty, h_\infty e_\infty\colon \mathscr{E}_0^> \cap \mathscr{H}_0^< \to A$-mod coincide and have image the category A-reg of regular A-modules.

(2) The restricted functor $e_\infty h_\infty = h_\infty e_\infty\colon \mathscr{E}_0^> \cap \mathscr{H}_0^< = C\text{-reg} \to A\text{-reg}$ is the functor $F\colon C\text{-reg} \to A\text{-reg}$, defined in [7].

We give a new description of the functors h_∞, e_∞, and F in terms of minimal approximations.

THEOREM. *Let $A = k\mathscr{Q}$ be a connected wild hereditary algebra and X a quasi-simple stone in A-reg.*

(a) *For a module $M \in \mathscr{H}_0^<$, the minimal right* $\operatorname{add}\{\tau_A X, \tau_C^i Z \mid i > 0\}$-*approximation $\rho\colon K(M) \to M$ is injective with cokernel $h_\infty(M)$.*

(b) *For an A-module $M \in \mathscr{E}_0^>$, the minimal left* $\operatorname{add}\{\tau_A X, \tau_C^{-i} Z \mid i > 0\}$-*approximation $\lambda\colon M \to Q(M)$ is surjective with kernel $e_\infty(M)$.*

(c) *For $M \in \mathscr{E}_0^> \cap \mathscr{H}_0^<$ the following diagram is commutative*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K(e_\infty(M)) \arrow[r] \arrow[d, equal] & e_\infty(M) \arrow[r] \arrow[d] & F(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K(M) \arrow[r] & M \arrow[r] \arrow[d] & h_\infty(M) \arrow[r] \arrow[d] & 0 \\
 & & Q(M) \arrow[r, equal] \arrow[d] & Q(h_\infty(M)) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Proof. (a) We have $h_\infty(M) = h_m(M)$ for some $m \gg 0$, that is $h_{m+i}(M) = h_m(M)$ for all $i \geq 0$. This is equivalent to $\operatorname{Hom}_A(\tau_A^{m+i+1}X, h_{m+i}(M)) = 0$ and consequently by [18, 4.4] $\operatorname{Hom}_A(\tau_C^{m+i}Z, M) = 0$ for all $i \geq 0$. Therefore the minimal right $\operatorname{add}(\tau_A X \oplus (\bigoplus_{i=1}^{m-1} \tau_C^i Z))$-approximation is a minimal right $\operatorname{add}\{\tau_A X, \tau_C^i Z \mid i > 0\}$-approximation, simultaneously. The claim follows from 1.6. Part (b) is dual.

(c) From (2) we get the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K(e_\infty(M)) \arrow[r] & e_\infty(M) \arrow[r] \arrow[d] & F(M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K(M) \arrow[r] & M \arrow[r] \arrow[d] & h_\infty(M) \arrow[r] \arrow[d] & 0 \\
 & & Q(M) \arrow[d] & Q(h_\infty(M)) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Since ρ: $K(M) \to M$ is a right approximation, there exists a map f: $K(e_\infty(M)) \to K(M)$, making the left square commutative. Similarly there is a g: $Q(M) \to Q(h_\infty(M))$, making the bottom square commutative. Let $K \in \{K(M), K(e_\infty(M))\}$ and $Q \in \{Q(M), Q(h_\infty(M))\}$. We have $\mathrm{Hom}_A(K, Q) = 0$, since the module K is in $\mathrm{add}\{\tau_A X, \tau_C^i Z \mid i > 0\}$ and Q is in $\mathrm{add}\{X, \tau_C^{-i} Z \mid i > 0\}$. Consequently we get $\mathrm{Hom}_A(K, e_\infty(M)) \cong \mathrm{Hom}_A(K, M)$ and $\mathrm{Hom}_A(M, Q) \cong \mathrm{Hom}_A(h_\infty(M), Q)$. Therefore f, g both are isomorphisms.

1.11 Since C-reg $= \mathscr{H}_0^{<} \cap \mathscr{E}_0^{>}$, we get from 1.7 and 1.9

COROLLARY. *Let $M \in C$-reg and n be any positive integer.*

(a) *The minimal right* add $\bigoplus_{1 \le i \le n} \tau_C^i Z$*-approximation* $\rho_{[1,n]}$: $\bigoplus_{1 \le i \le n} \tau_C^i Z^{\alpha_i} \to M$ *is injective.*

(b) *The minimal left* add $\bigoplus_{1 \le i \le n} \tau_C^{-i} Z$*-approximation* $\lambda_{[1,n]}$: $M \to \bigoplus_{1 \le i \le n} \tau_C^{-i} Z^{\beta_i}$ *is surjective.*

It is shown in the next section that any of these conditions implies that Z is orbital elementary in C-reg.

2. ORBITAL ELEMENTARY MODULES

2.1. Let $A = k\mathscr{Q}$ be a connected wild hereditary algebra. Following [22, 17], we call an indecomposable regular A-module E *elementary*, if there is no short exact sequence $0 \to U \to E \to V \to 0$ with U, V both nonzero and regular. For example, if E is an indecomposable regular module with minimal dimension, then it is elementary. Elementary modules always are quasi-simple bricks, [17]. The converse is not true.

We call an indecomposable regular module E *additive elementary*, if each short exact sequence $0 \to U \to E^m \to V \to 0$, with U, V regular and $m \geq 1$, splits. Examples of additively elementary modules are elementary stones, [15], or regular modules E, such that the one-point extension $A[E]$ is quasi-tilted, [9]. In [9] there also is shown that elementary modules do not need to be additively elementary.

An indecomposable regular module E is called *orbital elementary*, if each short exact sequence $0 \to U \to \tau_A^{i_1} E^{r_1} \oplus \cdots \oplus \tau_A^{i_m} E^{r_m} \to V \to 0$ with U, V regular and $\tau_A^{i_1} E^{r_1} \oplus \cdots \oplus \tau_A^{i_m} E^{r_m} \in \operatorname{add}\{\tau_A^i E | i \in \mathbb{Z}\}$, splits.

Since the Auslander−Reiten translation τ_A defines an equivalence τ_A: A-reg $\to A$-reg, a module E is (additively, orbital) elementary if and only if so is $\tau_A^i E$, for any integer i.

2.2. It was shown in [17, 1.2], that for a regular module M there exists an integer m such that the kernels of all morphisms g: $\tau_A^r M \to R$ for $r \geq m$ and R regular, are regular. The integer m only depends on $\dim_k M$.

Let M and N be regular A-modules and f: $M \to N$ a morphism. Consequently the kernel K of f is preprojective if and only if $\tau_A^r f$: $\tau_A^r M \to \tau_A^r N$ is injective for $r \gg 0$, by [17, 1.2]. Dually, the cokernel Q of f is preinjective, if and only if $\tau_A^{-r} f$: $\tau_A^{-r} M \to \tau_A^{-r} N$ is surjective for $r \gg 0$. This fact is used frequently in what follows.

We now give another characterization of orbital elementary modules. Section 1.11 then implies their existence.

PROPOSITION. *Let $A = k\mathcal{Q}$ be a connected wild hereditary algebra and E be a quasi-simple brick. The following are equivalent.*

(a) *E is orbital elementary.*

(b) *For any pair of integers $m \leq n$ and for any regular module R, the minimal right* add $\bigoplus_{i=m}^n \tau_A^i E$*-approximation $\rho_{[m,n]}$: $\bigoplus_{i=m}^n \tau_A^i E^{r_i} \to R$ has preprojective kernel.*

(c) *For any pair of integers $m \leq n$ and for any regular module R, the minimal left* add $\bigoplus_{i=m}^n \tau_A^i E$*-approximation $\lambda_{[m,n]}$: $R \to$* add $\bigoplus_{i=m}^n \tau_A^i E^{s_i}$ *has preinjective cokernel.*

Proof. (a) $\Rightarrow$ (b): Let $\rho_{[m,n]}$: $\bigoplus_{i=m}^n \tau_A^i E^{r_i} \to R$ be a minimal right add $\bigoplus_{i=m}^n \tau_A^i E$-approximation of R. By [17] there exists an $\ell > 0$, such that the shifted map $\tau_A^\ell \rho_{[m,n]}$: $\bigoplus_{i=m}^n \tau_A^{i+\lambda} E^{r_i} \to \tau_A^\ell R$ has regular kernel K. Hence we get the regular short exact sequence

$$0 \to K \to \bigoplus_{i=m}^n \tau_A^{i+\ell} E^{r_i} \to \operatorname{Im} \tau_A^\ell \rho[m,n] \to 0,$$

which splits, by (a). Since $\tau_A^\ell \rho_{[m,n]}$ is a minimal right add $\bigoplus_{i=m+\ell}^{n+\ell} \tau_A^i E$-approximation of $\tau_A^\ell R$, we get $K = 0$, by the definition of minimality.

Hence $\rho_{[m,n]}$ has preprojective kernel. The implication (a) ⇒ (c) is shown similarly.

(b) ⇒ (a): Assume there exists a nonsplit short exact sequence

$$0 \to U \xrightarrow{f} \bigoplus_{j=1}^{s} \tau_A^{n_j} E^{r_j} \xrightarrow{g} V \to 0,$$

with U, V regular, with $n_1 < n_2 < \cdots < n_s$ and $r_j > 0$. By [4] we may assume that f and g are minimal, that means no nonzero direct summand of $\overline{E} = \bigoplus_{j=1}^{s} \tau_A^{n_j} E^{r_j}$ is contained in the kernel of g and $\operatorname{Im} f$ is not contained in a proper direct summand of $\overline{E}$.

Let ρ: $\bigoplus_{j=n_1}^{n_s} \tau_A^j E^{t_j} = E_1 \to V$ be the minimal right add $\bigoplus_{j=n_1}^{n_s} \tau_A^j E$-approximation of V. Modulo some τ_A-shift, we may assume that ρ is injective, by (b). Since ρ is an approximation, we get the following commutative diagram

$$\begin{array}{ccccccc} U & \longrightarrow & \overline{E} & \xrightarrow{g} & V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \| & & \\ 0 & \longrightarrow & E_1 & \xrightarrow{\rho} & V & & \end{array} .$$

Therefore ρ is an isomorphism, and we identify V with E_1 via ρ.

Since E is a quasi-simple brick, we have $\operatorname{Hom}_A(E, \tau_A^{-i} E) = 0$, for all $i > 0$, [13]. Therefore g is of the form

$$\begin{pmatrix} g_0 & g' \\ 0 & g_1 \end{pmatrix} : \left(\bigoplus_{j=1}^{s-1} \tau_A^{n_j} E^{r_j} \right) \oplus \tau_A^{n_s} E^{r_s} \to \left(\bigoplus_{j=1}^{n_s - 1} \tau_A^j E^{t_j} \right) \oplus \tau_A^{n_s} E^{t_s},$$

where g_1: $\tau_A^{n_s} E^{r_s} \to \tau_A^{n_s} E^{t_s}$.

Since $\tau_A^{n_s} E$ is a brick, it follows from [24, 1.2], that $\ker g_1$ is a direct summand of $\tau_A^{n_s} E^{r_s}$. Since g is a right minimal map, g_1 is injective. Hence it is, again by [24, 1.2] a split mono. Take h_1: $\tau_A^{n_s} E^{t_s} \to \tau_A^{n_s} E^{r_s}$ with $g_1 h_1 = 1$. Then h_1 is surjective and $h_1 g_1 = \epsilon_1 \in \operatorname{End}_A(\tau_A^{n_s} E^{t_s})$ is an idempotent.

Define

$$m = \begin{pmatrix} 0 & 0 \\ 0 & m_1 \end{pmatrix} \left(\bigoplus_{j=1}^{n_s - 1} \tau_A^j E^{t_j} \right) \oplus \tau_A^{n_s} E^{t_s} \to \left(\bigoplus_{j=1}^{s-1} \tau_A^{n_j} E^{r_j} \right) \oplus \tau_A^{n_s} E^{r_s}.$$

The map m has image $\tau_A^{n_s} E^{r_s}$ and $mg \neq 0$ is an idempotent endomorphism of $\operatorname{End}_A(E_1)$. Hence $\operatorname{Im} f \subset \bigoplus_{j=1}^{s-1} \tau_A^{n_j} E^{r_j}$ which contradicts to the minimality of f and the claim follows.

Similarly one shows (c) $\Rightarrow$ (a).

2.3. We denote by $T_{2,3,m}$ with $m \geq 7$ the graph

$$\begin{array}{ccccccccc} & & & & \cdot & & & & \\ & & & & | & & & & \\ \cdot & - & \cdot & - & 1 & - & 2 & - \cdots - & m. \end{array}$$

THEOREM. *Let $A = k\mathcal{Q}$ be a connected wild hereditary algebra.*

(a) *If the underlying graph of $\mathcal{Q}$ is not of type $T_{2,3,m}$, then A has an orbital elementary module E with $\mathrm{End}_A(E) = k$.*

(b) *If $M \neq 0$ is a regular module, such that the one-point extension $A[M]$ is a tilted algebra of type $H = k\mathcal{Q}'$, then M is orbital elementary.*

Proof. (a) It was shown by Lache [19], that for a connected wild hereditary algebra $A = k\mathcal{Q}$, where $\mathcal{Q}$ is not of type $T_{2,3,m}$ there exists a pair (H, X), where H is a connected wild hereditary algebra and X is a quasi-simple regular H-module such that $X^{\perp} \cong C$-mod, where C is a connected wild hereditary algebra and A is concealed of type C. By 1.11 and 2.2 there exists an orbital elementary module Z in C-reg with endomorphism ring k. Since A is concealed of type C, the categories C-reg and A-reg are equivalent. Therefore there also exist orbital elementary A-module E with $\mathrm{End}_A(E) = k$.

(b) is similar to [17, 5.3]. If $A[M]$ is tilted, say of type H, then there exists a tilting module $T \in H$-mod with $\mathrm{End}_H(T) \cong A[M]$.

The tilting module T has a decomposition $T = T_0 \oplus X$, where X is quasi-simple regular in H-mod and T_0 is a preprojective tilting module in $X^{\perp}$. Let $0 \to \tau_H X \to Z \to X \to 0$ be the Auslander–Reiten sequence, ending in X. We know that Z is orbital elementary in $X^{\perp}$. We get

$$\mathrm{End}_H(T) = \begin{pmatrix} \mathrm{End}_H(T_0) & \mathrm{Hom}_H(T_0, Z) \\ 0 & k \end{pmatrix},$$

and $\mathrm{End}_H(T_0) \cong A$. Since the functor $\mathrm{Hom}_H(T_0, -)$ defines an equivalence from the category of regular $X^{\perp}$-modules to the category A-reg, the module $\mathrm{Hom}_H(T_0, Z)$ is orbital elementary in A-reg, too. By [17, 5.2] we finally have $M \cong_{\gamma} \mathrm{Hom}_H(T_0, Z)$ for some automorphism $\gamma \in \mathrm{Aut}(A)$. Consequently also M is orbital elementary.

2.4. The following example shows that not all additively elementary modules are orbital elementary. Let k be some elementary algebraically closed field and consider the quivers

$$
\begin{tikzcd}[row sep=small]
 & \overset{2}{\bullet} \arrow[dl] & & \\
\underset{1}{\bullet} & & \underset{3}{\bullet} \arrow[ll] \arrow[ul] & \underset{4}{\bullet} \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, shift left] \arrow[l, shift right] & \underset{3}{\bullet} \arrow[l]
\end{tikzcd}.
$$

$$\mathscr{Q}' \qquad\qquad\qquad \mathscr{Q}$$

Let $A = k\mathscr{Q}$ be the path algebra of the quiver $\mathscr{Q}$. By [22] an indecomposable regular A-module E is elementary, if and only if either $\underline{\dim}\, \tau_A^m E = (1,1,0)$ for some $m \in \mathbb{Z}$, or the indecomposable regular module U_1 with $\underline{\dim}\, U_1 = (1,2,0)$ is in the τ_A-orbit of E. Since U_1 has no self-extensions, it is additively elementary, [15]. Take X elementary with $\underline{\dim}\, X = (1,1,0)$. Let $U_2 = \tau_A U_1$, hence $\underline{\dim}\, U_2 = (3,4,4)$.

It is easy to check for $i = 1,2$ that $\dim \operatorname{Hom}_A(X, U_i) = 1$. Take $0 \neq f_i \in \operatorname{Hom}_A(X, U_i)$. Clearly f_i is injective. Consider the short exact sequence

$$(\xi) \quad 0 \to X \xrightarrow{(f_1, f_2)} U_1 \oplus U_2 \xrightarrow{(\pi_1, \pi_2)^t} Q \to 0,$$

where Q is the cokernel of $f = (f_1, f_2)$. One has $\underline{\dim}\, Q = (3,5,4) = \underline{\dim}\, \tau_A^2 X$.

If $\langle -, - \rangle$ denotes the homological bilinear form on the Grothendieck group of A, [25] one has $\langle \underline{\dim}(U_1 \oplus U_2), \underline{\dim}\, Q \rangle = 2$.

Since U_1 and U_2 are orthogonal stones and $\pi = (\pi_1, \pi_2)^t$ is surjective, it follows $\operatorname{Hom}_A(Q, U_1 \oplus U_2) = 0$ and consequently $\operatorname{Ext}_A^1(U_1, Q) = 0$.

Since $f_1\colon X \to U_1$ is injective with preinjective cokernel, the induced map $(f_1, \tau_A^2 U_1)\colon \operatorname{Hom}_A(U_1, \tau_A^2 U_1) \to \operatorname{Hom}_A(X, \tau_A^2 U_1)$ is injective. From $\operatorname{Hom}_A(U_2, \tau_A^2 U_1) = 0$, we get $(f_1, \tau_A^2 U_1) = (f, \tau_A^2 U_1)$. If $(-, \tau_A^2 U_1)$ stands for $\operatorname{Hom}_A(-, \tau_A^2 U_1)$, we get from (ξ):

$$0 \to \left(Q, \tau_A^2 U_1\right) \to \left(U_1 \oplus U_2, \tau_A^2 U_1\right) \xrightarrow{(f, \tau_A^2 U_1)} \left(X, \tau_A^2 U_1\right).$$

Since $(f, \tau_A^2 U_1)$ is injective, $\operatorname{Hom}_A(Q, \tau_A^2 U_1) = 0$ as follows. By the Auslander–Reiten formula, this means $\operatorname{Ext}_A^1(U_2, Q) = 0$, and $\dim \operatorname{Hom}_A(U_1 \oplus U_2, Q) = 2$ follows.

Consider now

$$0 \to \operatorname{End}_A(Q) \to \operatorname{Hom}_A(U_1 \oplus U_2, Q) \xrightarrow{(f, Q)} \operatorname{Hom}_A(X, Q).$$

Since $(f, Q) \neq 0$, we get $1 \leq \dim \operatorname{End}_A(Q) < \dim \operatorname{Hom}_A(U_1 \oplus U_2, Q) = 2$. Therefore Q is a brick. Since $\underline{\dim}\, Q = (3,5,4)$ it is an elementary module, especially it is regular. Hence U_1 is not orbital elementary.

Let $H = k\mathscr{Q}'$ and P be the minimal projective generator of $(\tau_H^+ S_2)^\perp$. Then $T = P \oplus \tau_H^- S_2$ is a tilting module with $\operatorname{End}_H(T) \cong A[E]$, where E is any elementary module with $\underline{\dim} E = (1,1,0)$. Consequently E is orbital elementary.

2.5. If E is an orbital elementary A-module and ρ: $\bigoplus_{i=0}^{m} \tau_A^i E^{r_i} \to R$ is a minimal right add$(\bigoplus_{i=0}^{m} \tau_A^i E)$-approximation of some regular module R, then the kernel $\ker \rho$ is preprojective. Consequently, there exists some s, such that $\tau_A^s \rho$ is injective. By [17, 1.2], the integer s can be chosen independently of R, only depending on $\dim(\bigoplus_{i=0}^{m} \tau_A^i E)$. The module Z not only is orbital elementary in C-mod, it has the additional property, that for each regular C-module R and for any natural number $m \geq 1$ the minimal right add$(\bigoplus_{i=1}^{m} \tau_C^i A)$-approximation ρ: $\bigoplus_{i=1}^{m} \tau_C^i Z^{r_i} \to R$ is injective. Dually is the minimal left add$(\bigoplus_{i=1}^{m} \tau_C^{-i} Z)$-approximation λ: $R \to \bigoplus_{i=1}^{m} \tau_C^{-i} Z^{t_i}$ surjective for all regular modules R.

3. THE ORBIT ALGEBRA OF A REGULAR MODULE

Let $A = k\mathcal{Q}$ be a connected wild hereditary algebra. We denote by Φ the Coxeter transformation on $K_0(A)$, that is the linear map defined by the property $\Phi(\underline{\dim} X) = \underline{\dim}\, \tau_A X$ for all nonprojective indecomposable A-modules. The spectral radius ρ of Φ, also called the growth-number of A is bigger than 1 in the wild case, see, for example, [8].

3.1. The *orbit algebra* $\mathcal{O}(X)$ of a regular A-module X is a $\mathbb{Z}$-graded algebra $\mathcal{O}(X) = \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_i(X)$ with $\mathcal{O}_i(X) = \mathrm{Hom}_A(X, \tau^i X)$ and where multiplication is defined by the following rule: For $f \in \mathcal{O}_i(X)$ and $g \in \mathcal{O}_j(X)$ the product fg is the composition $f \cdot \tau^i g \in \mathcal{O}_{i+j}(X)$, [20]. In order to avoid ambiguity, we use the symbol $\cdot$ for the composition of maps, if necessary.

Since τ is an equivalence on A-reg, we have $\mathcal{O}(X) \cong \mathcal{O}(\tau_A X)$ for a regular module X, so the orbit algebra really is an object attached to the τ-orbit X^τ of X.

LEMMA. *Let $X \neq 0$ be a regular A-module.*

(a) $\mathcal{O}_m(X) = 0$ *for* $m \ll 0$.

(b) $\lim_{m \to \infty} \rho^{-m} \dim \mathcal{O}_m(X) = \alpha > 0$.

Proof. (a) follows from [12] and (b) from [23] and [26].

3.2. LEMMA. *Let be f_i: $U_i \to V_i$, $i = 1, 2$ nonzero maps between regular modules. Then there exists a homomorphism h: $V_1 \to \tau^s U_2$ for some s with* $f_1 \cdot h \cdot \tau^s f_2 \neq 0$.

Proof. Choose a number $t \geq 0$ with $\ker(\tau_A^t f_2)$ regular, [17, 1.2]. By [12] and [21] we find some number $s > t$ with

(1) $\operatorname{Hom}_A(\tau_A^{s-1}U_2, V_1) = 0 = \operatorname{Hom}_A(\ker(\tau_A^{s-1}f_2), V_1)$,

(2) there exists an injective homomorphism $g\colon V_1 \to \operatorname{Im}(\tau_A^s f_2)$.

If K and Q denote the kernel and the image of $\tau_A^s f_2$, we get the regular short exact sequence $0 \to K \to \tau_A^s U_2 \xrightarrow{\pi} Q \to 0$. If $\epsilon\colon Q \to \tau_A^s V_2$ is the embedding, then $\pi \cdot \epsilon = \tau_A^s f_2$ holds.

Applying the functor $\operatorname{Hom}_A(V_1, -)$ to this short sequence, we get

$$\operatorname{Hom}_A(V_1, \tau_A^s U_2) \xrightarrow{(V_1, \pi)} \operatorname{Hom}_A(V_1, Q) \to \operatorname{Ext}_A^1(V_1, K).$$

But $\operatorname{Ext}_A^1(V_1, K) = 0$ by condition (1), thus (V_1, π) is surjective, that is $g = h \cdot \pi$ for some $h \in \operatorname{Hom}_A(V_1, \tau_A^s U_1)$. So we get $f_1 \cdot h \cdot \tau_A^s f_2 = f_1 \cdot h \cdot \pi \cdot \epsilon = f_1 \cdot g \cdot \epsilon \neq 0$ since $f_1 \neq 0$ and $g \cdot \epsilon$ is injective.

3.3. PROPOSITION. *Let $0 \neq X$ be a regular A-module.*

(a) *$\mathscr{O}(X)$ is a prime ring.*

(b) *$\mathscr{O}(X)$ is a domain, if and only if X is an elementary module.*

(c) *For $m \gg 0$ there are $x, y \in \mathscr{O}_m(X)$, such that the k-subalgebra $k\langle x,y \rangle$ of $\mathscr{O}(X)$, generated by $1, x, y$ is a free algebra in x, y.*

Proof. (a) For $f, g \in \mathscr{O}(X) \setminus \{0\}$ we have to find an $h \in \mathscr{O}(X)$ with $fhg \neq 0$. Since $\mathscr{O}(X)$ is $\mathbb{Z}$-graded, we can choose f, g, and h homogeneous. But in this case the claim follows from 3.2.

(b) Let X be elementary. Then $\tau_A^i X$ is elementary for all $i \in \mathbb{Z}$. Let $0 \neq f_i \in \mathscr{O}_{n_i}(X)$, for $i = 1, 2$. Since the cokernel of f_1 is preinjective and the kernel of $\tau_A^{n_1} f_2$ is preprojective by [17, 1.3], the product $f_1 f_2 \in \mathscr{O}(X)$ is nonzero. Since $\mathbb{Z}$ is linearly ordered, $\mathscr{O}(X)$ therefore is a domain.

Assume X is not elementary. So we have nonzero regular modules U and V and a short exact sequence

$$0 \to U \xrightarrow{\epsilon} X \xrightarrow{\pi} V \to 0.$$

Take i such that there is a nonzero morphism $f_1 \in \operatorname{Hom}(X, \tau^i U)$ and j such that a nonzero map $f_2 \in \operatorname{Hom}(\tau^i V, \tau^{i+j} X)$ exists, [6]. Then $(f_1 \cdot \tau^i \epsilon)(\pi \cdot \tau^{-i} f_2) = 0$ but both factors are nonzero.

(c) By [21, 2.3], for $m \gg 0$ there exists a mono $(x, y)^t\colon X^2 \to \tau_A^m X$. It is straightforward, to check that the algebra $k\langle x, y \rangle$ is a free algebra.

3.4. Since τ: A-reg $\to$ A-reg is an equivalence, one also can consider for a regular module $X \neq 0$ the algebra

$$\overline{\mathscr{O}}(X) = \bigoplus_{i \in \mathbb{Z}} \overline{\mathscr{O}}_i(X),$$

with $\overline{\mathscr{O}}_i(X) = \operatorname{Hom}(\tau^{-i}X, X)$ and multiplication $fg = (\tau^{-i}f) \cdot g$ for $f \in \overline{\mathscr{O}}_i(X)$ and $g \in \overline{\mathscr{O}}_j(X)$. One easily checks that the assignment $\bar{\tau}$: $\overline{\mathscr{O}}(X) \to \mathscr{O}(X)$, defined by $\bar{\tau}(f) = \tau^i f$ for $f \in \overline{\mathscr{O}}_i(X)$ defines an isomorphism of $\mathbb{Z}$-graded algebras. We use this fact in what follows for dual formulations.

4. FINITELY GENERATED ORBIT ALGEBRAS

4.1. LEMMA. *For a regular A-modules $X \neq 0$ the following properties are equivalent.*

(a) *There exists a short exact sequence*

$$0 \to R \xrightarrow{f} \tau_A^{i_1} X^{r_1} \oplus \cdots \oplus \tau_A^{i_m} X^{r_m} \xrightarrow{g} \tau_A^n X \to 0,$$

with $i_1 < i_2 < \cdots < i_m < n$ and R regular.

(b) *The minimal right* add $\bigoplus_{j=i_1}^{n-1} \tau_A^j X$*-approximation of $\tau_A^n X$ is surjective with regular kernel K.*

Proof. (b) $\Rightarrow$ (a) is clear, we show (a) $\Rightarrow$ (b): Let ρ: $\bigoplus_{j=i_1}^{n-1} \tau_A^j X^{s_j} \to \tau_A^n X$ be the minimal right add $\bigoplus_{j=i_1}^{n-1} \tau_A^j X$-approximation of $\tau_A^n X$. Then g factors through ρ, so we get the commutative diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \longrightarrow & \bigoplus_{t=1}^m \tau_A^{i_t} X^{r_t} & \xrightarrow{g} & \tau_A^n X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \| & & \\
0 & \longrightarrow & K & \longrightarrow & \bigoplus_{t=i_1}^{n-1} \tau_A^t X^{s_t} & \xrightarrow{\rho} & \tau_A^n X & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Q & = & Q & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} .$$

Since g is surjective, so is ρ. Since K is a submodule of a regular module, it has no nonzero preinjective direct summand. Dually Q has no preprojective direct summand. Since K is an extension of $(R)f'$ by Q and $(R)f'$ is regular, the module K is regular, too.

4.2. We are interested in the case, when $\mathscr{O}(X)$ is a finitely generated k-algebra. It is clear that $\mathscr{O}(X)$ is finitely generated if and only if there exists a finite generator system, consisting in homogeneous elements. Let $\mathscr{O}_+(X) = \bigoplus_{i>0} \mathscr{O}_i(X)$. It is a subalgebra of $\mathscr{O}(X)$ without identity.

LEMMA. *$\mathscr{O}(X)$ is a finitely generated algebra, if $\mathscr{O}_+(X)$ is finitely generated.*

Proof. $\bigoplus_{i \le 0} \mathscr{O}_i(X)$ is finite dimensional by 3.1(a).

4.3. PROPOSITION. *Let X be a regular A-module. The following are equivalent.*

(a) *$\mathscr{O}_+(X)$ is a finitely generated k-algebra.*

(b) *There exists some epimorphism ϕ: $\tau_A^{i_1} X^{r_1} \oplus \cdots \oplus \tau_A^{i_s} X^{r_s} \to \tau_A^n X$ where $i_1 < i_2 < \cdots < i_s < n$ with regular kernel K.*

(c) *There exists some monomorphism ψ: $\tau_A^m X \to \tau_A^{i_1} X^{r_1} \oplus \cdots \oplus \tau_A^{i_s} X^{r_s}$ where $m < i_1 < i_2 < \cdots < i_s$ with regular cokernel Q.*

Proof. (b) ⇒ (a): By some τ_A-shift, we may assume $i_1 \ge 0$.

Let $\phi = (\phi_1^{(1)}, \dots, \phi_1^{(r_1)}, \dots, \phi_s^{(1)}, \dots, \phi_s^{(r_s)})^t$: $\overline{X} = \tau_A^{i_1} X^{r_1} \oplus \cdots \oplus \tau_A^{i_s} X^{r_s} \to \tau_A^n X$, with $\phi_t^{(j)}$: $\tau_A^{i_t} X \to \tau_A^n X$, for all $1 \le j \le r_j$. Take $m > 0$, such that $\operatorname{Ext}_A^1(\tau_A^{-s} X, K) = 0$, for all $s \ge m$, [12]. Application of $\operatorname{Hom}_A(\tau_A^{-s} X, -)$ to the short exact sequence

$$0 \to K \to \overline{X} \xrightarrow{\phi} \tau_A^n X \to 0$$

gives $\operatorname{Hom}_A(\tau_A^{-s} X, \overline{X}) \xrightarrow{(\tau_A^{-s} X, \phi)} \operatorname{Hom}_A(\tau_A^{-s} X, \tau_A^n X) \to 0$.

Therefore each homomorphism f: $\tau_A^{-s} X \to \tau_A^n X$ factors through ϕ. In terms of the orbit algebra $\mathscr{O}_+(X)$ this means $\tau_A^s f = \sum_{i=1}^{s} \sum_{j=1}^{r_i} f_i^{(j)} \phi_i^{(j)}$ with $f_i^{(j)}$: $X \to \tau_A^{s+r_i} X$. Therefore $\mathscr{O}_+(X)$ is generated by the finite-dimensional subspace $\bigoplus_{i=1}^{m+n-1} \mathscr{O}_i(X)$. The implication (c) ⇒ (a) is shown similarly, see 3.4.

(a) ⇒ (c): Let $\mathscr{G} = (f_{i_1}^{(1)}, \dots, f_{i_1}^{(r_1)}, \dots, f_{i_s}^{(1)}, \dots, f_{i_s}^{(r_s)})$, with $f_{i_m}^{(j)} \in \mathscr{O}_{i_m}(X)$, be a finite generator-system of $\mathscr{O}_+(X)$ and let $\phi = (f_{i_1}^{(1)}, \dots, f_{i_s}^{(r_s)})$: $X \to \overline{X} = \bigoplus_{t=1}^{s} \tau_A^{i_t} X^{r_t}$ be the induced map. We may assume modulo some shift in τ_A^--direction, that the cokernel Q of ϕ is regular, [17, 1.2]. Let I be the image of ϕ and K be its kernel. Let $\phi = \pi \cdot \epsilon$ with π: $X \to I$ the projection and ϵ: $I \to \overline{X}$ the inclusion. Take f: $X \to \tau_A^n X$ with $n > i_s$. Since $\mathscr{G}$ is a generator-system of $\mathscr{O}_+(X)$, we have $f = \sum_{t,j} f_{i_t}^{(j)} g_{i_t}^{(j)}$ for suitable $g_{i_t}^{(j)} \in \mathscr{O}_{n-i_t}(X)$, which means $f = \phi \cdot \bar{g} = \pi \cdot (\epsilon \cdot \bar{g})$. Hence each f: $X \to \tau_A^n X$ factors through π.

Assume $K \ne 0$. Then $K = K_1 \oplus K_2$ with K_1 preprojective and K_2 regular. For $m \gg 0$ we get $\operatorname{Hom}_A(K, \tau_A^m X) \ne 0$, by [6], but $\operatorname{Ext}_A^1(I, \tau_A^m X) = 0$, by [12]. Application of $\operatorname{Hom}_A(-, \tau_A^m X)$ to the short exact sequence $0 \to K \to X \xrightarrow{\pi} I \to 0$ gives

$$\operatorname{Hom}_A(I, \tau_A^m X) \xrightarrow{(\pi, \tau^m X)} \operatorname{Hom}_A(X, \tau_A^m X) \xrightarrow{\alpha} \operatorname{Hom}_A(K, \tau_A^m X) \to 0.$$

Let $g \in \operatorname{Hom}_A(X, \tau_A^m X)$ with $\alpha(g) \neq 0$. Then g does not have a factorization through π, a contradiction. Hence we have $K = 0$. Similar is (a) $\Rightarrow$ (b), see 3.4.

Note that the parts (b), respectively, (c) of the proposition can be formulated in terms of minimal right, respectively, left approximations, by 4.1.

4.4. COROLLARY. *If E is an orbital elementary module, then $\mathscr{O}(E)$ is not finitely generated.*

4.5. EXAMPLE. Let T be a regular partial tilting module with $\tau_A^i T$ sincere, for all integers i. For example, this is the case, when T is a regular tilting module. We show that $\mathscr{O}(T)$ is finitely generated. Since a regular partial tilting module T with $\tau_A^i T$ sincere for all i is not an elementary module [17, 3.4], the orbit algebra $\mathscr{O}(T)$ has zero divisors in this case, by 3.3.

By [14, 4.7] there exists some positive integer $t(A)$, depending only on A, with the property that for any two regular modules X, Y with $\operatorname{Hom}_A(X, Y) \neq 0$, we have $\operatorname{Hom}_A(X, \tau_A^m Y) \neq 0$ for $m \geq t(A)$. Let $m \geq t(A)$ and let $f_1, \dots, f_r$ be a k-basis of $\operatorname{Hom}_A(T, \tau_A^m T)$. Let $\lambda = (f_1, \dots, f_r)$: $T \to \tau_A^m T^r$. By some shift in τ_A^--direction, we may assume that the cokernel Q of λ is regular, [17, 1.2]. The claim follows from 4.3, if λ is injective. Let K be the kernel of λ and I the image. Since I is cogenerated by $\tau_A^m T$ and $\operatorname{Ext}_A^1(\tau_A^m T, \tau_A^m T) = 0$, we get $\operatorname{Ext}_A^1(I, \tau_A^m T) = 0$. Application of $\operatorname{Hom}_A(-, \tau_A^m T)$ to the short exact sequence $0 \to K \to T \xrightarrow{\pi} I \to 0$ gives $0 \to \operatorname{Hom}_A(I, \tau_A^m T) \to \operatorname{Hom}_A(T, \tau_A^m T) \to \operatorname{Hom}_A(K, \tau_A^m T) \to 0$. Since $(\pi, \tau_A^m T)$: $\operatorname{Hom}_A(I, \tau_A^m T) \to \operatorname{Hom}_A(T, \tau_A^m T)$ is an isomorphism by the definition of λ we get $\operatorname{Hom}_A(K, \tau_A^m T) = 0$. Since $\tau_A^i T$ is sincere, for all integers i, the module K has no nonzero preprojective direct summand. It has no nonzero regular direct summand, by the choice of $m \geq t(A)$. Hence $K = 0$, and the claim follows from 4.3.

5. CANONICAL GENERATING SYSTEMS

The main result in this part is a criterium when the orbit algebra $\mathscr{O}(X)$ is a free algebra as a $\mathbb{Z}$-graded algebra. This criterium implies, that the orbit algebra of an orbital elementary module has this property.

5.1. If X is a regular A-module with $\operatorname{Hom}_A(X, \tau_A^{-i} X) = 0$ for $i > 0$, then the minimal add $\bigoplus_{i=0}^r \tau_A^i X$-approximations of regular modules can be constructed recursively.

LEMMA. *Let X be a module with* $\operatorname{Hom}_A(X, \tau_A^{-i} X) = 0$ *for all* $i > 0$. *Let* $0 \leq r \leq s$ *be natural numbers, R a regular module and let* ρ_1: $X^{a_r} \oplus \tau_A X^{a_{r-1}}$

$\oplus \cdots \oplus \tau_A^r X^{a_0} \to R$ *be a minimal right* add $\oplus_{0 \le i \le r} \tau_A^i X$*-approximation of* R *and* ρ_2: $X^{b_s} \oplus \tau_A X^{b_{s-1}} \oplus \cdots \oplus \tau_A^s X^{b_0} \to \tau_A^{s-r} R$ *a minimal right* add $\oplus_{0 \le i \le s} \tau_A^i X$*-approximation of* $\tau_A^{s-r}R$. *Then we have* $a_i = b_i$ *for* $0 \le i \le r$.

Proof. Let $\rho_2 = (\sigma_1, \sigma_2)^t$: $(\oplus_{0 \le i \le s-r-1} \tau_A^i X^{b_i}) \oplus (\oplus_{s-r \le i \le s} \tau_A^i X^{b_i}) \to \tau_A^{s-r}R$. Take $Y \in$ add $\oplus_{s-r \le i \le s} \tau_A^i X$ and a map f: $Y \to \tau_A^{s-r}R$. Since ρ_2 is an approximation, we have $f = \alpha \cdot \rho = \alpha_1 \cdot \sigma_1 + \alpha_2 \sigma_2$. But α_1: $Y \to \oplus_{0 \le i \le s-r-1} \tau_A^i X^{b_i}$ is zero by the assumption on X, hence σ_2 is a right add $\oplus_{s-r \le i \le s} \tau_A^i X$-approximation of $\tau_A^{s-r}R$. Clearly σ_2 is a minimal map. Hence $\sigma_2 = \tau^{s-r}\rho_1$, and the claim follows.

5.2. Let X be a quasi-simple brick with $\mathrm{End}_A(X) = k$. We call $(g_0 = 1, g_1, \dots, g_m)$ a *homogeneous generating system of* $\oplus_{0 \le i \le n} \mathscr{O}_i$, if g_i is in some $\mathscr{O}_j$, with $0 \le j \le n$ and $\oplus_{0 \le i \le n} \mathscr{O}_i$ is contained in the k-subalgebra $k\langle g_i | 0 \le i \le m\rangle$ of $\mathscr{O}(X)$, generated by the g_i.

LEMMA. *Let X be a quasi-simple brick with* $\mathrm{End}_A(X) = k$. *There are equivalent.*

(a) *Let* $m \ge 0$. *The set* $\mathscr{F} = \{f_0 = 1, f_i^{(j)} | 1 \le i \le m, 1 \le j \le r_i\}$ *with* $0 \le r_i$ *and* $f_i^{(j)} \in \mathscr{O}_i(X)$ *is a minimal homogeneous generating system of* $\oplus_{0 \le i \le m} \mathscr{O}_i(X)$.

(b) *The map* $\lambda = (f_1^{(1)}, \dots, f_1^{(r_1)}, \dots, f_m^{(1)}, \dots, f_m^{(rm)})$: $X \to \oplus_{1 \le i \le m} \tau_A^i X^{r_i}$ *is a minimal left* add $\oplus_{i=1}^m \tau_A^i X$*-approximation of* X.

(c) *Let* $g_i^{(j)} = \tau_A^{m-i} f_i^{(j)}$: $\tau_A^{m-i} X \to \tau_A^m X$.

The map $\rho = (g_m^{(1)}, \dots, g_m^{(r_m)}, \dots, g_1^{(1)}, \dots, g_1^{(r_1)})^t$: $\oplus_{i=0}^{m-1} \tau_A^i X^{r_{m-i}} \to \tau_A^m X$ *is a minimal right* add $\oplus_{i=0}^{m-1} \tau_A^i X$*-approximation of* $\tau_A^m X$.

Proof. (a) ⇒ (b): Since $\mathscr{F}$ is a minimal generating system, the map λ is a minimal map.

Let ℓ: $X \to \overline{X}$ be a minimal left add $\oplus_{i=1}^m \tau_A^i X$-approximation of X. Then there exists an α: $\overline{X} \to Y = \oplus_{i=1}^m \tau_A^i X^{r_i}$ with $\lambda = \ell \cdot \alpha$. Since $\mathscr{F}$ is a generating system of $\oplus_{i=0}^m \mathscr{O}_i(X)$, each component of ℓ is of the form $\sum_{i,j} f_i^{(j)} \cdot \beta_i^{(j)}$ for some homogeneous elements $\beta_i^{(j)} \in \mathscr{O}(X)$. Let β: $Y \to \overline{X}$ be the map, defined by the $\beta_i^{(j)}$. Then we have $\ell = \lambda \cdot \beta$. Since ℓ is minimal, we get from $\ell = \ell \cdot \alpha \cdot \beta$, that $\alpha \cdot \beta \in \mathrm{Aut}(\overline{X})$. Similarly, we get from the minimality of λ that $\beta \cdot \alpha \in \mathrm{Aut}(Y)$. Therefore λ is a minimal left approximation.

(b) ⇒ (a): Consider a minimal left add $\oplus_{i=1}^m \tau_A^i X$-approximation

$$\lambda = \left(f_1^{(1)}, \dots, f_1^{(s_1)}, \dots, f_m^{(1)}, \dots, f_m^{(s_m)}\right): X \to \overline{X} = \oplus_{i=1}^m \tau_A^i X^{s_i},$$

of X and let $g \in \bigoplus_{i=0}^{m} \mathscr{O}_i(X)$. We claim that g is contained in the subalgebra $k\langle f_0 = 1, f_i^{(j)}\rangle$ of $\mathscr{O}(X)$, generated by $\{f_0, f_i^{(j)}\}$. It is enough to consider g homogeneous, say, $g \in \mathscr{O}_i(X)$, with $i \geq 0$. We prove the claim by induction on i, the case $i = 0$ is trivial. Let $0 < i \leq m$. Since λ is a left approximation, we have $g = \sum_{r=1}^{m} \sum_{j=1}^{s_r} f_r^{(j)} \cdot h_r^{(j)}$, with $h_r^{(j)}\colon \tau_A^r X \to \tau_A^i X$. Since X is a quasi-simple brick, we have $\operatorname{Hom}_A(\tau_A^r X, \tau_A^i X) \neq 0$ only if $r \leq i$ [13, 1.2]. Therefore by induction we get $\tau_A^{-r} h_r^{(j)} \in k\langle f_0, f_{m'}^{(j)}, m' < i\rangle$, which proves the claim.

The proof of the equivalence (a) $\Leftrightarrow$ (c) follows from (a) $\Leftrightarrow$ (b) by 4.3.

5.3. THEOREM. *Let $A = k\mathscr{Q}$ be a connected wild hereditary algebra and X be a quasi-simple brick with* $\operatorname{End}_A(X) = k$. *The following are equivalent.*

(a) *$\mathscr{O}(X)$ is free, as a $\mathbb{Z}$-graded k-algebra.*

(b) *For each $n \geq 1$, the kernel K of the minimal right* add $\bigoplus_{0 \leq i < n} \tau_A^i X$*-approximation $\rho_{(0,n-1)}\colon X_{(0,n-1)} = \bigoplus_{0 \leq i < n} \tau_A^i X^{r_i} \to \tau_A^n X$ is preprojective.*

(c) *For each $n \geq 1$, the cokernel Q of the minimal left* add $\bigoplus_{0 < i \leq n} \tau_A^i X$*-approximation $\lambda_{(1,n)}\colon X \to X_{(1,n)} = \bigoplus_{0 < i \leq n} \tau_A^i X^{r_i}$ is preinjective.*

If $\mathscr{O}(X)$ is free, it is not finitely generated.

Proof. (a) $\Rightarrow$ (c): Let $\mathscr{O}(X) = k\langle X_i^{(j)} | 1 \leq i,\ 1 \leq j \leq r_i\rangle$ be a free algebra in the noncommuting letters $X_i^{(j)} \in \mathscr{O}_i(X)$. By 5.2, the induced map

$$\lambda_{(1,n)} = \left(X_1^{(1)}, \dots, X_1^{(r_1)}, \dots, X_n^{(1)}, \dots, X_n^{(r_n)}\right)\colon X \to \bigoplus_{0 < i \leq n} \tau_A^i X^{r_i}$$

is a minimal left approximation. Modulo some shift in τ_A^--direction, the cokernel Q of $\lambda_{(1,n)}$ is regular, [17]. We have to show that $Q = 0$ in this case. Assume $Q \neq 0$ and consider the exact sequence

$$X \xrightarrow{\lambda_{(1,n)}} X_{(1,n)} \xrightarrow{\pi} Q \to 0,$$

with $\pi = (\pi_1^{(1)}, \dots, \pi_N^{(r_n)})^t$. By Lukas [21, 2.3] there is for $m \gg 0$ a mono $\alpha\colon\ Q \to \tau_A^m X$. We get $0 = \lambda_{(1,n)} \cdot \pi \cdot \alpha = \sum_{i=1}^{n} \sum_{j=1}^{r_i} X_i^{(j)} \cdot (\pi_i^{(j)} \cdot \alpha) = \sum_{i,j} X_i^{(j)} \tau_A^{-i}(\pi_i^{(j)} \cdot \alpha)$. Since $\mathscr{O}(X)$ is a free algebra in the letters $X_i^{(j)}$, this implies $\tau_A^{-i}(\pi_i^{(j)} \cdot \alpha) = 0$ for all i, j, hence $\pi_i^{(j)} \cdot \alpha = 0$ for all i, j. Since α is injective, this means $\pi = 0$, a contradiction since $Q \neq 0$.

The proof (a) $\Rightarrow$ (b) is similar, see 4.3.

(b) $\Rightarrow$ (a): Take for $n \geq 1$ a minimal right add $\bigoplus_{0 \leq i < n} \tau_A^i X$-approximation

$$\rho_{(0,n-1)} = \left(\rho_n^{(1)}, \dots, \rho_n^{(r_n)}, \dots, \rho_1^{(1)}, \dots, \rho_1^{(r_1)}\right)^t\colon$$

$$X_{(0,n-1)} = \bigoplus_{0 \leq i < n} \tau_A^i X^{r_{n-i}} \to \tau_A^n X,$$

with $\rho_i^{(j)} \in \operatorname{Hom}_A(\tau_A^{n-i}X, \tau_A^n X)$. Let $X_i^{(j)} = \tau_A^{i-n}\rho_i^{(j)} \in \mathscr{O}_i(X)$, for all i, j. We show, by induction on n that the set

$$\mathscr{X}_n = \left\{ X_{i_1}^{(j_1)} \cdots X_{i_t}^{(j_t)} \,\middle|\, \sum_{m=1}^{t} i_m = n, 1 \le j_m \le r_m \right\},$$

of different words in the $X_i^{(j)}$ of weight $n = \sum i_m$ forms a k-basis of $\mathscr{O}_n(X)$. Note that we use 5.1.

It is clear for $n = 1$, assume therefore $n > 1$. We know from 5.2, that $\mathscr{X}_n$ generates the k-vector space $\mathscr{O}_n(X)$. Let $\sum_{X_{i_1}^{(j_1)} \cdots X_{i_r}^{(j_t)} \in \mathscr{X}_n} z_{i_1 \cdots i_t}^{(j_1 \cdots j_t)} X_{i_1}^{(j_1)} \cdots X_{i_t}^{(j_t)} = 0$.

Ordering by the last letter, we get $\sum_{1 \le i_t \le n} \sum_{1 \le j_t \le r_{j_t}} f_{i_t}^{(j_t)} X_{i_t}^{(j_t)} = 0$ where $f_{i_t}^{(j_t)} = \sum_{i_1 + \cdots + i_{t-1} = n - i_t} \sum_{j_1 \cdots j_{t-1}} z_{i_1 \cdots i_t}^{(j_1 \cdots j_t)} X_{i_1}^{(j_1)} \cdots X_{i_{t-1}}^{(j_{t-1})} \in \mathscr{O}_{n-i_t}(X)$.

Therefore $(f_n^{(1)}, \ldots, f_n^{(r_n)}, \ldots, f_1^{(1)}, \ldots, f_1^{(r_1)})\colon X \to \bigoplus_{i=0}^{n-1} \tau_A^i X^{r_{n-1}}$ has as image the kernel of $\rho_{(0,n-1)}$. Since this kernel is preprojective, we have $f_i^{(j)} = 0$ for all i, j. Hence $f_n^{(j)} = z_n^{(j)} = 0$, for $1 \le j \le r_n$. For $i < n$, $f_i^{(j)}$ is a linear combination of words in $\mathscr{X}_{n-1}$. Therefore also in this case the coefficients $z_{i_1 \cdots i_{t-1}, i}^{(j_1 \cdots j_{t-1}, j)}$, with $\sum_{j=1}^{t-1} i_j = n - i$ are zero by induction and the claim follows.

(c) $\Rightarrow$ (a) is similar. The last statement follows from 4.1, 4.3.

5.4. COROLLARY. *If E is an orbital elementary module with* $\operatorname{End}_A(E) = k$, *then the orbit algebra $\mathscr{O}(E)$ is a free k-algebra in infinitely many variables.*

5.5. The following lemma might be well known. I learned it from H. Strauss. Since the proof is surprisingly short and easy, it is given.

LEMMA. *Let X be a regular module. Then $\sum_{i \ge 0} \langle \underline{\dim} X, \underline{\dim}\, \tau_A^i X \rangle t^i$ is a rational function.*

Proof. Let be $x = \underline{\dim} X$, Φ the Coxeter transformation corresponding to τ, that is $\Phi x = \underline{\dim}\, \tau X$ and $\chi_\Phi(t)$ the characteristic polynomial of Φ, say $\chi_\Phi(t) = \sum_{r=0}^n a_r t_r$. Then we have

$$\chi_\Phi(t) \sum_{i \ge 0} \langle x, \Phi^i x \rangle t^i = \sum_{l \ge 0} r_l t^l,$$

with $r_l = \langle x, \sum_{i=0}^{\min\{l,n\}} a_i \Phi^{l-i} x \rangle$. From the Cayley–Hamilton theorem we get $r_l = 0$ for $l \ge n$, thus the series $\sum \langle x, \Phi^i x \rangle t^i$ is a rational function.

5.6. COROLLARY. *Let $X \ne 0$ be regular such that the orbit algebra $\mathscr{O}(X)$ is a free $\mathbb{Z}$-graded algebra, say $\mathscr{O}(X) = k\langle X_i^{(j)} | i > 0,\ 1 \le j \le r_i \rangle$, with $X_i^{(j)} \in \mathscr{O}_i(X)$. Then $\sum_{i>0} r_i t^i$ is a rational function.*

Proof. Since $\mathscr{O}(X)$ is a free algebra, it follows from Theorem 5.3, that for each $n > 0$ the module $\tau_A^n X$ has a minimal right add $\oplus_{i=0}^{n-1} \tau_A^i X$-approximation

$$\rho: \oplus_{i=0}^{n-1} \tau_A^i X^{r_{n-i}} \to \tau_A^n X,$$

with preprojective kernel K. The map (X, ρ) is surjective, since ρ is a right approximation. It is injective, since K is preprojective. Therefore we have $\mathrm{Hom}_A(X, \tau_A^n X) \cong \mathrm{Hom}_A(X, \oplus_{i=0}^{n-1} \tau_A^i X^{r_{n-i}})$.

Let $a_i = \dim \mathrm{Hom}_A(X, \tau_A^i X)$. We have $a_0 = 1$, $a_1 = r_1$, and $a_n = \sum_{i=1}^n r_i a_{n-i}$, for $n \geq 1$. From this we get $\sum_{i \geq 1} a_i t^i = \sum_{i \geq 1}(\sum_{j=1}^i r_j a_{i-j}) t^i = (\sum_{j \geq 0} a_j t^j)(\sum_{i \geq 1} r_i t^i)$. Since $a_i = \langle \underline{\dim} X, \underline{\dim}\, \tau_A^i X \rangle$ for $i > 1$, we get from 5.5, that $f(t) = \sum_{i \geq 1} a_i t^i$ is a rational function. Therefore $\sum_{i \geq 1} r_i t^i = f(t)(1 + f(t))^{-1}$ is a rational function, too.

5.7. Example. Let $A = k\mathscr{Q}$ be a connected wild hereditary algebra with an orbital elementary module E' with $\mathrm{Ext}_A^1(E', E') = 0$. Take an integer s such that $\tau_A^{-s-t} E'$ is sincere for all $t \geq 0$ and let E be $\tau_A^{-s} E'$. Since E is faithful, the minimal left approximation $f: A \to E^r$ is injective. It is easy to check that the cokernel V of f is regular and that $T = E \oplus V$ is a regular tilting module. Moreover it follows from [10] that $\tau_A^i V$ is sincere for all integers i. Therefore the orbit algebras $\mathscr{O}(T)$ and $\mathscr{O}(V)$ are finitely generated by 4.5, whereas $\mathscr{O}(E)$ is a free and not finitely generated algebra. Let $e \in \mathrm{End}_A(T)$ be the projection on E with kernel V and $f = 1_T - e$. Then we get $e\mathscr{O}(T)e \cong \mathscr{O}(E)$, $f\mathscr{O}(T)f \cong \mathscr{O}(V)$ and

$$\mathscr{O}(T) \cong \begin{pmatrix} \mathscr{O}(E) & e\mathscr{O}(T)f \\ f\mathscr{O}(T)e & \mathscr{O}(V) \end{pmatrix}.$$

5.8. Let k be an algebraically closed field, $A = k\mathscr{Q}$ wild hereditary, and E an elementary module, such that $\tau_A^i E$ has no nontrivial regular factor modules, for all $i \geq 0$. Take $0 \neq f: E \to \tau_A^m E$ for some $m > 0$. Then all maps $\tau_A^i f: \tau_A^i E \to \tau_A^{m+i} E$ are injective, for $i \geq 0$. In [21] in this situation the infinite-dimensional A-module

$$D(E, f) = \varinjlim \left(E \xrightarrow{f} \tau_A^m E \xrightarrow{\tau^m f} \tau_A^{2m} E \xrightarrow{\tau^{2m} f} \cdots \to \tau_A^{rm} E \to \cdots \right)$$

was studied, and it was shown that all nonzero endomorphisms of $D(E, f)$ were injective. Therefore $D(E, f)$ is indecomposable.

Assume now, that E additionally is orbital elementary with orbit algebra $\mathscr{O}(E) = k\langle X_i^{(j)} | i \geq 1, 1 \leq j \leq r_i \rangle$.

PROPOSITION. *Let E be orbital elementary such that $\tau^i E$ has no nontrivial regular factor modules and $m \geq 1$ such that $r_m > 0$. Then the following holds.*

(a) $\operatorname{End}_A(D(E, X_m^{(i)})) = k$.

(b) *For* $1 \leq i \neq j \leq r_m$ *we get* $\operatorname{Hom}_A(D(E, X_m^{(i)}), D(E, X_m^{(j)})) = 0$.

Proof. Define $X = X_m^{(i)}$, $Y = X_m^{(j)}$, $D = D(E, X)$, and $D' = D(E, Y)$. We may consider all the modules $\tau_A^{rm} E$ as submodules of D, respectively, D' and we get $D = \bigcup_{r \geq 0} \tau_A^{rm} E$, respectively, $D' = \bigcup_{r \geq 0} \tau_A^{rm} E$ in this case. If $f \colon D \to D$, respectively, $f \colon D \to D'$ is a morphism and $r \geq 0$ a natural number, $(\tau_a^{rm} E)f$ is a finitely generated submodule of D, respectively, D', hence it is a submodule of $\tau_A^{\bar{r}m} E$, for some $\bar{r}$. Let r' be minimal with $(\tau_A^{rm} E)f \subset \tau_A^{(r+r')m} E$. If $f \neq 0$, then f is injective by [21], and we get $r' \geq 0$, by [13]. We denote by $f_r \colon \tau_A^{rm} E \to \tau_A^{(r+r')m} E$ the restriction of f.

(a) Take $0 \neq f \in \operatorname{End}_A(D)$ and $r \geq 0$. For $t \geq r + r'$ we get the commutative square

$$\begin{tikzcd}
\tau^{rm}E \arrow[r, "\tau^{rm}X"] \arrow[d, "f_r"] & \tau^{(r+1)m}E \arrow[r] & \cdots \; \tau^{tm}E \arrow[d, "f_t"] \\
\tau^{(r+r')m}E \arrow[r, "\tau^{(r+r')m}X"] & \tau^{(r+r'+1)m}E \arrow[r] & \cdots \; \tau^{(t+t')m}E
\end{tikzcd} .$$

In terms of the orbit algebra $\mathscr{O}(E)$ this reads $(X_m^{(i)})^{t-r} f_t = f_r (X_m^{(i)})^{t+t'-r-r'}$. Since $\mathscr{O}(E)$ is a free algebra, we get $f_r = a(X_m^{(i)})^{r'}$ and $f_t = a(X_m^{(i)})^{t'}$ for some $a \in k^*$.

Since r', t' were chosen minimal with respect to the property $(\tau_A^{rm} E)f \subset \tau_A^{(r+r')m} E$, respectively, $(\tau_A^{tm} E)f \subset \tau_A^{(t+t')m} E$, it follows $r' = t' = 0$, that is, $f = a$.

(b) Assume there exists $0 \neq f \colon D \to D'$. Take again $t \geq r + r'$ as in part (a) and consider the commutative diagram

$$\begin{tikzcd}
\tau^{rm}E \arrow[r, "\tau^{rm}X"] \arrow[d, "f_r"] & \tau^{(r+1)m}E \arrow[r] & \cdots \; \tau^{tm}E \arrow[d, "f_t"] \\
\tau^{(r+r')m}E \arrow[r, "\tau^{(r+r')m}Y"] & \tau^{(r+r'+1)m}E \arrow[r] & \cdots \; \tau^{(t+t')m}E
\end{tikzcd} .$$

which reads in $\mathscr{O}(E)$ as $(X_m^{(i)})^{t-r} f_t = f_t (X_m^{(j)})^{t+t'-r-r'}$. For $t \gg 0$ this implies $f_r = f_t = 0$, hence $f = 0$, a contradiction.

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